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LETTER TO THE EDITOR

Site and bond directed branched polymers for arbitrary dimensionality: evidence supporting a relation with the Lee–Yang edge singularity

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Abstract. Parisi and Sourlas have argued that the isotropic branched polymer problem in dimension $d+1$ is related to the Lee–Yang edge singularity problem in dimension $d-1$. To test if there is a relation with the *directed* branched polymer problem, we calculate the generating functions for both site and bond directed branched polymers for *arbitrary* dimension d to order $s_{max}=10$ and $b_{max}=8$. Our analysis lends support to the proposal that $\theta(d+1)-1=\theta_D(d)=\sigma_{LY}(d-1)+1$, where θ and θ_D are the critical exponents for lattice animals and directed lattice animals, and σ_{LY} is the Lee–Yang edge singularity exponent. We also obtain expansions for the growth parameter in the variable $\sigma^{-1}=(2d-1)^{-1}$ for bond and site directed lattice animals.

The statistics of the dilute limit of branched polymers in a good solvent are well described by the *isotropic* lattice animal problem. The related problem of *directed* branched polymers, or directed lattice animals, has recently received attention, in part due to (i) their anisotropic shape, which must be described by two independent diverging lengths (Redner and Yang 1982, Day and Lubensky 1982), (ii) the possibility of an exact result for the directed site lattice animal (DSLA) problem on the two-dimensional square and triangular lattices (Dhar *et al* 1982, Nadal *et al* 1982), and (iii) the intriguing fact that a field-theory formulation (Day and Lubensky 1982) of the directed polymer problem (with loops neglected) shows striking parallels with the ARB_2 branched polymer model near their respective upper marginal dimensions. For these reasons, it is desirable to obtain further information on both site and bond directed lattice animals for as many spatial dimensionalities as possible.

Here we calculate the number $A_s(d)$ of DSLA for a hypercubic lattice of *arbitrary* dimension d for animals of up to ten sites. For the directed *bond* lattice animal (DBLA) problem, we obtain general-d expressions for the number $A_b(d)$ for animals of up to $b_{max}=8$ bonds. This latter work extends recent calculations for $A_b(d)$ for specific values of d (Redner and Yang 1982) to higher order. We analyse the series to obtain estimates of the critical parameters for $2\leqslant d\leqslant 8$ for both the DSLA and DBLA problems.

The DSLA enumeration proceeds by a generalisation of the Martin algorithm (Martin 1974) using a variation of the computer program presented by Redner (1982).

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Table 1. The coefficients W_{sk} appearing in equation (1) for the directed site lattice animal problem and the coefficients W_{bk} appearing in equation (3) for the directed-bond lattice animal problem in d dimensions.

(*a*)

s \ k	1	2	3	4	5	6	7	8	9
2	1								
3	3	1							
4	16	11	1						
5	125	135	33	1					
6	1296	1944	828	94	1				
7	16 807	32 585	20 405	4541	265	1			
8	262 144	626 688	525 760	182 816	23 764	748	1		
9	4782 969	13 640 319	14 460 444	6992 010	1514 584	121 881	2121	1	
10	100 000 000	332 000 000	427 350 000	267 734 500	83 826 650	12 030 404	620 211	6044	1

(*b*)

b \ k	0	1	2	3	4	5	6	7
1	1							
2	3	1						
3	16	12	1					
4	125	150	40	1				
5	1296	2160	1032	128	1			
6	16 807	36 015	25 304	6384	410	1		
7	262 144	688 128	643 430	253 628	37 863	1324	1	
8	4782 969	14 880 348	17 441 529	9481 320	2354 814	220 875	4316	1

We find that the functions $A_s(d)$ are of the same form as for the isotropic lattice animal problem (Gaunt *et al* 1976, Gaunt and Ruskin 1978, Gaunt 1980)

$$A_1(d)=1 \qquad A_s(d)=\sum_{k=1}^{s-1} W_{sk}\binom{d}{s-k} \qquad (s\geqslant 2). \tag{1}$$

The (d-independent) coefficients W_{sk} are given in table 1(a) up to $s=10$†. From the exact Cayley tree results, we expect

$$W_{s1}=s^{s-2}, \tag{2a}$$

and this expectation is confirmed by the first column. The numbers W_{s2} and W_{s3} in the second and third columns can also be calculated. We find

$$W_{s2}=s^{s-4}[\tfrac{1}{2}(s-2)(s^2-2s+3)] \tag{2b}$$

and

$$W_{s3}=s^{s-6}[\tfrac{1}{24}(s-3)(3s^5-20s^4+54s^3-91s^2+198s-360)]. \tag{2c}$$

The DBLA enumeration proceeds as in Redner and Yang (1982), and the functions $A_b(d)$ have the same general-d form as for the isotropic case (Gaunt 1980):

$$A_b(d)=\sum_{k=0}^{b-1} W_{bk}\binom{d}{b-k} \qquad (b\geqslant 1). \tag{3}$$

The coefficients W_{bk} are given in table 1(b). From the Cayley tree solution, we expect

$$W_{b0}=(b+1)^{b-1}. \tag{4a}$$

This expectation is borne out by the first column of table 1(b). The coefficients W_{b1} of the first column can also be calculated, with the result

$$W_{b1}=(b+1)^{b-3}[\tfrac{1}{2}b(b+1)(b-1)]. \tag{4b}$$

We find that the functions $A_s(d)$ and $A_b(d)$ have the same qualitative form as in the isotropic lattice animal problem:

$$A_s(d)\sim\lambda_{\rm D}^{s}s^{-\theta_{\rm D}} \qquad A_b(d)\sim\lambda_{\rm D}^{b}b^{-\theta_{\rm D}} \tag{5}$$

but with different growth parameters $\lambda_{\rm D}$ and a different exponent $\theta_{\rm D}$. Hence the corresponding generating functions

$$G_{\rm D}^{\rm site}(K)=\sum_{s=1}^{\infty} A_sK^s \qquad G_{\rm D}^{\rm bond}(K)=\sum_{b=1}^{\infty} A_bK^b \tag{6}$$

have the asymptotic forms

$$G_{\rm D}^{\rm site}(K)\sim(K-K_{\rm c}^{\rm site})^{\theta_{\rm D}-1} \qquad G_{\rm D}^{\rm bond}(K)\sim(K-K_{\rm c}^{\rm bond})^{\theta_{\rm D}-1}. \tag{7}$$

Here the 'critical fugacity' $K_{\rm c}$ is the reciprocal of the growth parameter $\lambda_{\rm D}$.

† We obtained coefficients beyond $s=10$ for small values of d: for $d=3$, we found $A_{11}=3201\,967$, $A_{12}=16\,164\,384$, $A_{13}=82\,044\,151$, $A_{14}=418\,352\,107$ and $A_{15}=2\,141\,761\,669$, while for $d=4$, $A_{11}=106\,048\,124$ and $A_{12}=779\,247\,701$ and for $d=5$ $A_{11}=1\,443\,787\,863$. Dhar (private communication) also independently calculated A_1–A_{15} for $d=3$.

Table 2. Dependence on d of relevant critical parameters defined in the text. The figures in parentheses give the error in the last digit(s).

d	1	2	3	4	5	6	7
$\theta_D^{site}(d)$	0[c]	0.500(2)	0.82(5)	1.04(6)	1.17(8)	1.27(10)	1.5[c]
$\theta_D^{bond}(d)$	0[c]	0.515(15)	0.92(5)	1.20(10)	1.33(10)	1.39(10)	1.5[c]
$\theta(d+1)-1$	0[c]	0.50(5)[a]	0.90(7)[a]	1.1(1)[a]	1.3(2)[a]	1.4(2)[a]	1.5[c]
$\sigma_{LY}(d-1)+1$	0[c]	0.5[c]	0.837(3)[b]	1.086(15)[b]	—	—	1.5[c]
$\theta_D^{Flory}(d)$	0[c]	0.5625	0.9	1.125	1.2857	1.40 625	1.5[c]
$\nu_\perp^{site}$	—	0.500(2)	0.410(25)	0.35(2)	0.29(2)	0.25(2)	0.25[c]
$\nu_\perp^{bond}$	—	0.515(15)	0.460(25)	0.40(3)	0.33(0)	0.28(2)	0.25[c]
$\nu_\perp^{Flory}$	—	0.5625	0.45	0.375	0.32	0.28	0.25
λ_D^{site}	1	3.000(2)	5.410(10)	8.01(4)	10.68(8)	13.40(10)	16.15(12)
λ_D^{bond}	1	3.508(8)	6.305(15)	9.13(5)	11.90(10)	14.70(15)	17.24(30)

[a] Gaunt (1980), Gaunt and Ruskin (1978), Gaunt *et al* (1976).
[b] Kurtze and Fisher (1979), Kortman and Griffiths (1971).
[c] Exact result.

The generating functions $G^{site}(K)$ and $G^{bond}(K)$ were analysed for $2 \leq d \leq 8$ using standard methods†, and the results for the growth parameter λ_D and the exponent θ_D are given in table 2, along with the corresponding values of $\theta(d+1)$ for the isotropic animal problem, and $\sigma_{LY}(d-1)$ for the Lee–Yang edge singularity problem. The agreement supports the proposal (figure 1) that

$$\theta(d+1)-1=\theta_D(d)=\sigma_{LY}(d-1)+1. \tag{8}$$

Table 2 also lists values of $\nu_\perp$, given by the relation $\theta_D=(d-1)\nu_\perp$ which was obtained by Family (1982) using the Ginzburg criterion. Here $\nu_\perp$ governs the increase in the animal radius transverse to the 'time' axis. Also tabulated are values of $\theta_D^{Flory}=9(d-1)/[4(d+2)]$ obtained by Family (1982) from the Flory expression $\nu_\perp^{Flory}=9/[4(d+2)]$ (Redner and Coniglio 1982, Lubensky and Vannimenus 1982).

Although less reliable than direct analysis, an approximation to the growth parameter may be obtained by deriving expansions in the variable σ^{-1}, where $\sigma=2d-1$.

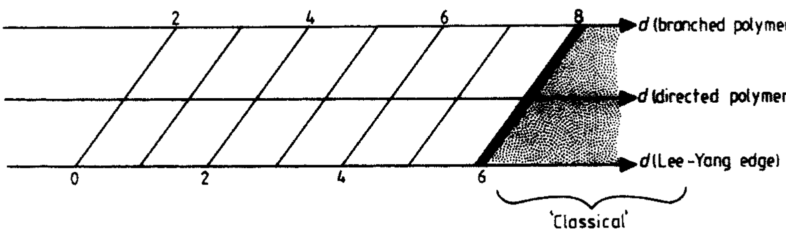

Figure 1. Schematic illustration of the proposed correspondence, equation (8), between the branched polymer problem in dimension $d+1$, the Lee–Yang edge singularity problem in dimension $d-1$, and the directed branched polymer problem in dimension d.

† In particular, Padé analysis of the first and second derivatives of the series yield extremely accurate estimates of the critical parameters (particularly in low d). For higher dimensions alternative methods appear to be more reliable. There appears to be a systematic effect that the site series are better converged in low dimensions, while the bond series are better converged in higher dimensions. We calculated the coefficients in the directed bond problem using *site* counting, but found no improvement in the convergence.

Following the procedure used for isotropic lattice animals (Gaunt *et al* 1976, Gaunt and Ruskin 1978), we find

$$\ln \lambda_{\mathrm{D}}^{\mathrm{site}}(d) = \ln(\tfrac{1}{2}\sigma) + 1 - \sigma^{-1} - \tfrac{5}{3}\sigma^{-2} - \mathrm{O}(\sigma^{-3}) \qquad (9a)$$

and

$$\ln \lambda_{\mathrm{D}}^{\mathrm{bond}}(d) = \ln(\tfrac{1}{2}\sigma) + 1 + \mathrm{O}(\sigma^{-2}). \qquad (9b)$$

For large d, these expressions agree well with the estimates in table 2 for λ_{D}.

In summary, we have calculated the coefficients in the generating functions for the site and bond directed branched polymer problems up to order $s_{\max} = 10$ and $b_{\max} = 8$ for arbitrary dimensionality d. Our analysis supports the proposed relation (8) between the exponents of the directed problem and the corresponding exponents of the isotropic branched polymer problem for dimension $d+1$ and the Lee–Yang edge singular for dimension $d-1$. It would be desirable to develop a theory that supports the proposal, especially since there is already a field-theory argument supporting the equality $\theta(d+1) - 1 = \sigma_{\mathrm{LY}}(d-1) + 1$ (Parisi and Sourlas 1981). Of possible relevance to (8) is a recent preprint (Dhar 1982) mapping the DSLA problem for $d = 2$ onto the Baxter hard-square-lattice gas with anisotropic second-neighbour interaction and *negative* fugacity (complex 'magnetic field').

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Note added in proof. After this work was completed J L Cardy (private communication) informed us of a field-theory argument supporting equation (8).

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